PHYSICAL REVIEW E **77**, 061125 (2008)

Fractal probability laws

Iddo Eliazar[1,*] and Joseph Klafter[2,†]
[1]Department of Technology Management, Holon Institute of Technology, P.O. Box 305, Holon 58102, Israel
[2]School of Chemistry, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
(Received 22 February 2008; revised manuscript received 25 April 2008; published 18 June 2008)

We explore six classes of *fractal* probability laws defined on the positive half-line: Weibull, Frechét, Lévy, hyper Pareto, hyper beta, and hyper shot noise. Each of these classes admits a unique statistical power-law structure, and is uniquely associated with a certain operation of renormalization. All six classes turn out to be one-dimensional projections of underlying Poisson processes which, in turn, are the unique fixed points of Poissonian renormalizations. The first three classes correspond to *linear* Poissonian renormalizations and are intimately related to extreme value theory (Weibull, Frechét) and to the central limit theorem (Lévy). The other three classes correspond to *nonlinear* Poissonian renormalizations. Pareto's law—commonly perceived as the "universal fractal probability distribution"—is merely a special case of the hyper Pareto class.

DOI: 10.1103/PhysRevE.77.061125 PACS number(s): 02.50.−r, 05.45.Df

I. INTRODUCTION

Pareto's law—a statistical distribution describing a power-law connection between positive-valued measurements and their occurrence frequencies—is commonly perceived as the "universal fractal statistical distribution." This approach is masterfully described and advocated by Benoit Mandelbrot in a chapter titled "Scaling and power laws without geometry" appearing in his seminal monograph [1].

Pareto's law was discovered at the close of the nineteenth century by the Italian economist Vilfredo Pareto, while studying the statistics of human income [2]. Thereafter, Pareto's law was observed in empirical data originating from diverse scientific fields [3–6] (see also the review [7], and references therein).

Theoretically, the perception of Pareto's law as the "universal fractal statistical distribution" is based on two fundamentals: (i) the association of fractals with power laws, and (ii) the characterization of statistical laws by their survival probabilities.

In [8] the first fundamental "fractals=power laws"—in the context of random collections of real-valued data points—is examined and shown to be false. Indeed, introducing a definition of fractality based on the elemental notion of *geometric scale invariance*—rather than on the algebraic notion of power laws—leads to random fractal structures which are not necessarily governed by power laws.

In this paper we present the main findings of a recent research which questions the second fundamental [9]. While the survival probability characterizes statistical laws, it is not a *unique* characteristic. Rather, there are various probabilistic functionals—e.g., cumulative distribution function, hazard rate, cumulant sequence—which characterize statistical laws. So what happens when setting other characterizing probabilistic functionals to be power laws?

This question is explored in [9] by considering the eight most commonly practiced probabilistic functionals, categorized into the four following classes: *distribution functionals*: cumulative distribution function and survival probability function; *hazard functionals*: forward hazard rate and backward hazard rate (hazard rates play a key role in issues concerning the reliability of systems [10]); *Laplace functionals*: Laplace transform and log-Laplace transform; and *moment functionals*: moment sequence and cumulant sequence. Analyzing systematically these eight probabilistic functionals we arrive at six classes of probability laws, defined on the positive half-line, admitting an inherent power-law structure: *Weibull*, *Fréchet*, *Lévy*, *hyper Pareto*, *hyper beta*, and *hyper shot noise*. Pareto's law, commonly perceived as the "universal fractal probability distribution," is merely a special case of one of these six classes—the hyper Pareto class.

Each of the six classes is uniquely distinguished—among all probability distributions defined on the positive half-line—by the specific characterizing probabilistic functional set to be a power law. And, each of the first five classes turns out to uniquely manifest invariance under a specific type of an associated "probabilistic renormalization."

Moreover, all six classes turn out to possess an intrinsic *Poissonian structure*, and reflect a deeper notion of fractality present on the underlying Poissonian level. Indeed, all six classes are one-dimensional projections of underlying Poisson processes (defined on the positive half-line). And, these corresponding Poisson processes, in turn, are the unique fixed points of *Poissonian renormalizations*.

The first three classes—Weibull, Fréchet, and Lévy—are linked to *linear* Poissonian renormalizations and are categorized "linear." The other three classes—hyper Pareto, hyper beta, and hyper shot noise—are linked to *nonlinear* Poissonian renormalizations and are categorized "nonlinear."

The first three classes emerge from *extreme value theory* [11–13] and from the *central limit theorem* [14–16] as the universal linear scaling limits of minima (Weibull), maxima (Fréchet), and sums (Lévy) of sequences of independent and identically distributed positive-valued random variables (see also [17] for a more recent account). The aforementioned linearity or nonlinearity categorization is precisely what places the former three classes under the realm of extreme value theory and the central limit theorem—while leaving the latter three classes beyond their reach.

*eliazar@post.tau.ac.il
†klafter@post.tau.ac.il

 ©2008 The American Physical Society

The intrinsic Poissonian structure appearing both in this research and in [8] is elemental and of key importance. It connects together—in a deep and fundamental way, and via the class of *Paretian Poisson processes*—the Fréchet, Lévy, and Pareto laws discussed herein [18], and, it facilitates the definition of *Lorenzian fractality* in the context of random collections of positive-valued data points [19].

The remainder of this paper is organized as follows. We begin with some preliminaries in Sec. II, describe the *linear* classes of fractal probability laws in Sec. III, and the *nonlinear* classes in Sec. IV. For proofs of the results and assertions presented herein, the readers are referred to [9].

II. PRELIMINARIES

In this section we tersely review the notion of Poisson processes [20], and the notion of Poissonian renormalizations [8]. In this section and hereafter, the acronym IID stands for "independent and identically distributed."

A. Poisson processes

A Poisson process Π with rate function $r(x)$ $(x>0)$ is a countable collection of points scattered randomly on the positive half-line $(0,\infty)$, characterized by the following pair of properties [20]: (i) the number of points residing within the interval I is a Poisson-distributed random variable with mean $\int_I r(x)dx$, and (ii) the number of points residing within disjoint intervals are independent random variables.

Informally, the Poisson process Π places a single point within the infinitesimal interval $(x,x+dx)$ with probability $r(x)dx$, and leaves the infinitesimal interval $(x,x+dx)$ empty with probability $1-r(x)dx$—independently of all other infinitesimal intervals.

The *cumulative rate function* $\mathcal{R}_{\le}(x)$ of the Poisson process Π is the mean number of points residing within the interval $(0,x]$, and is given by $\mathcal{R}_{\le}(x)=\int_0^x r(x')dx'$. Contrarily, the *survival rate function* $\mathcal{R}_{>}(x)$ of the Poisson process Π is the mean number of points residing within the ray (x,∞), and is given by $\mathcal{R}_{>}(x)=\int_x^\infty r(x')dx'$.

For example, in the case of a homogeneous Poisson process with intensity λ $(\lambda>0)$ we have $r(x)\equiv\lambda$, the cumulative rate function is given by $\mathcal{R}_{\le}(x)=\lambda x$, and the survival rate function diverges $\mathcal{R}_{>}(x)\equiv\infty$.

In the sequel we shall make use of the three following functionals of a given Poisson process Π: (i) its *minimal point* $\min_{p\in\Pi}\{p\}$—which is well defined if and only if the rate function $r(x)$ is integrable at the origin; (ii) its *maximal point* $\max_{p\in\Pi}\{p\}$—which is well defined if and only if the rate function $r(x)$ is integrable at infinity; and (iii) its *point aggregate* $\Sigma_{p\in\Pi}p$—which, due to Campbell's theorem of the theory of Poisson processes ([20], Sec. 3.2), is convergent if and only if the rate function $r(x)$ satisfies the integrability condition $\int_0^\infty \min\{x,1\}r(x)dx<\infty$.

B. Poissonian renormalizations

Given a Poisson process Π with rate function $r(x)$ $(x>0)$ we construct its k-order *Poissonian renormalization* Π_k—k being a positive parameter—via the following two-step algorithm [8]: (i) replace the Poisson process Π by an intermediate Poisson process Π_k^{int} with rate function $r_k^{\text{int}}(x)=kr(x)$ $(x>0)$; (ii) shift each point of the intermediate process Π_k^{int} using a k-order monotone-increasing scaling function $\phi_k(x)$ $(x>0)$. The resulting k-order Poissonian renormalization is given by

$$\Pi_k=\{\phi_k(p)\}_{p\in\Pi_k^{\text{int}}}. \tag{1}$$

(When k is an integer then the intermediate Poisson process Π_k^{int} is the *union* of k IID copies of the Poisson process Π.)

The renormalization procedure is required to be *consistent*: A k-order renormalization followed by an l-order renormalization need equal a kl-order renormalization. This requirement implies the following *consistency condition,* which the scaling functions need satisfy: $\phi_k\circ\phi_l=\phi_{kl}$ $(k,l>0$; the sign $\circ$ denoting composition).

The two most fundamental Poissonian renormalizations are

Linear. A linear Poissonian renormalization is based on a set of linear scaling functions. The consistency condition implies that the linear scaling functions admit the form $\phi_k(x)=k^\varepsilon x$ $(x>0)$, where the exponent ε is an arbitrary nonzero parameter.

Power law. A power-law Poissonian renormalization is based on a set of power-law scaling functions. The consistency condition implies that the power-law scaling functions admit the form $\phi_k(x)=x^{k^\varepsilon}$ $(x>0)$, where the exponent ε is an arbitrary nonzero parameter.

The *fixed points* of Poissonian renormalizations are defined as follows [8]: A Poisson process Π is a fixed point of a given Poissonian renormalization if its renormalizations $\{\Pi_k\}_{k>0}$ are all equal in law.

III. LINEAR FRACTAL PROBABILITY LAWS

In this section we focus on the three *linear* classes of fractal probability laws defined on the positive half-line: Weibull, Fréchet, and Lévy. These fractal probability laws are one-dimensional projections of underlying Poissonian processes, which are the fixed points of linear Poissonian renormalizations—a fact rendering them the classification "linear." This linearity, in turn, is precisely why these fractal distributions emerge—in the context of *extreme value theory* [11–13], and in the context of the *central limit theorem* [14–16]—as the universal linear scaling limits of minima, maxima, and sums of sequences of IID positive-valued random variables.

A. Weibull

Definition. The Weibull law is defined on the positive half-line $(0,\infty)$ and governed by the survival probability function

$$\mathcal{P}_{>}(x)=\exp\{-ax^\alpha\} \tag{2}$$

$(x\ge 0$; the coefficient a and the exponent α being positive parameters).

Power-law structure. Among the class of probability distributions defined on the positive half-line, the Weibull law is the only probability distribution characterized by a power-law *forward hazard rate*. The forward hazard rate $\vec{H}(x)$ ($x>0$) of a positive-valued random variable X is defined as the limit $\vec{H}(x)=\lim_{\delta\to 0}\delta^{-1}\Pr(X<x+\delta|X\geq x)$ [10]. Informally, the forward hazard rate $\vec{H}(x)$ is the probability that the random variable X be realized at the point x—given that it was not realized within the interval $(0,x)$.

Probabilistic renormalization. As asserted by extreme value theory [11–13], the Weibull law is the *universal stochastic linear scaling limit* of *minima* of sequences of IID positive-valued random variables. This universality is manifested by the following "min-renormalization" characterization of the Weibull law:

Let $\{X_i\}_{i=1}^{\infty}$ be IID copies of a positive-valued random variable X, and consider the scaled minima $\wedge_n = n^{1/\alpha}\min\{X_1,\ldots,X_n\}$ ($n=1,2,\ldots$). Then, the scaled minima $\{\wedge_n\}_{n=1}^{\infty}$ are all equal in law if and only if the random variable X is governed by the survival probability function of Eq. (2).

Poissonian structure. The Weibull law is the probability distribution of the *minima* of Poisson processes defined on the positive half-line and governed by power-law *cumulative rate functions*. More specifically, a Poisson process Π is governed by the power-law cumulative rate function $\mathcal{R}_{\leq}(x)=ax^{\alpha}$ ($x>0$) if and only if its *minimal point* $\min_{p\in\Pi}\{p\}$ is governed by the survival probability function of Eq. (2).

Poissonian renormalization. Poisson processes defined on the positive half-line and governed by power-law cumulative rate functions constitute the fixed points of *linear Poissonian renormalizations* with positive exponents. More specifically, a Poisson process Π defined on the positive half-line is a fixed point of a linear Poissonian renormalization with scaling functions $\phi_k(x)=k^{1/\alpha}x$ if and only if its cumulative rate function admits the power-law form $\mathcal{R}_{\leq}(x)=ax^{\alpha}$ ($x>0$).

B. Fréchet

Definition. The Fréchet law is defined on the positive half-line $(0,\infty)$ and governed by the cumulative distribution function

$$\mathcal{P}_{\leq}(x)=\exp\{-ax^{-\alpha}\} \tag{3}$$

($x\geq 0$, the coefficient a and the exponent α being positive parameters).

Power-law structure. Among the class of probability distributions defined on the positive half-line the Fréchet law is the only probability distribution characterized by a power-law *backward hazard rate*. The backward hazard rate $\overleftarrow{H}(x)$ ($x>0$) of a positive-valued random variable X is defined as the limit $\overleftarrow{H}(x)=\lim_{\delta\to 0}\delta^{-1}\Pr(X>x-\delta|X\leq x)$ [10]. Informally, the backward hazard rate $\overleftarrow{H}(x)$ is the probability that the random variable X be realized at the point x—given that it was not realized within the ray (x,∞).

Probabilistic renormalization. As asserted by extreme value theory [11–13], the Fréchet law is the *universal stochastic linear scaling limit* of *maxima* of sequences of IID positive-valued random variables. This universality is manifested by the following "max-renormalization" characterization of the Fréchet law:

Let $\{X_i\}_{i=1}^{\infty}$ be IID copies of a positive-valued random variable X, and consider the scaled maxima $\vee_n = n^{-1/\alpha}\max\{X_1,\ldots X_n\}$ ($n=1,2,\ldots$). Then, the scaled maxima $\{\vee_n\}_{n=1}^{\infty}$ are all equal in law if and only if the random variable X is governed by the cumulative distribution function of Eq. (3).

Poissonian structure. The Fréchet law is the probability distribution of the *maxima* of Poisson processes defined on the positive half-line and governed by power-law *survival rate functions*. More specifically, a Poisson process Π is governed by the power-law survival rate function $\mathcal{R}_{>}(x)=ax^{-\alpha}$ ($x>0$) if and only if its *maximal point* $\max_{p\in\Pi}\{p\}$ is governed by the cumulative distribution function of Eq. (3).

Poissonian renormalization. Poisson processes defined on the positive half-line and governed by power-law survival rate functions constitute the fixed points of *linear Poissonian renormalizations* with negative exponents. More specifically, a Poisson process Π defined on the positive half-line is a fixed point of a linear Poissonian renormalization with scaling functions $\phi_k(x)=k^{-1/\alpha}x$ if and only if its survival rate function admits the power-law form $\mathcal{R}_{>}(x)=ax^{-\alpha}$ ($x>0$).

C. Lévy

Definition. The (one-sided) Lévy law is defined on the positive half-line $(0,\infty)$ and governed by a probability density function with Laplace transform

$$\mathcal{L}(\theta)=\exp\{-\Gamma(1-\alpha)a\theta^{\alpha}\} \tag{4}$$

($\theta\geq 0$, the coefficient a being a positive parameter, and the exponent α being in the range $0<\alpha<1$).

Power-law structure. Among the class of probability distributions defined on the positive half-line the Lévy law is the only probability distribution characterized by a power-law *log-Laplace transform*.

Probabilistic renormalization. As asserted by the central limit theorem [14–16], the Lévy law is the *universal stochastic linear scaling limit* of *sums* of sequences of IID positive-valued random variables. This universality is manifested by the following "sum-renormalization" characterization of the Lévy law:

Let $\{X_i\}_{i=1}^{\infty}$ be IID copies of a positive-valued random variable X, and consider the scaled sum $S_n=n^{-1/\alpha}\{X_1+\cdots+X_n\}$ ($n=1,2,\ldots$). Then, the scaled sums $\{S_n\}_{n=1}^{\infty}$ are all equal in law if and only if the random variable X is governed by the Laplace transform of Eq. (4).

Poissonian structure. The Lévy law is the probability distribution of the *point aggregates* of Poisson processes defined on the positive half-line and governed by power-law *survival rate functions*. More specifically, a Poisson process Π is governed by the power-law survival rate function $\mathcal{R}_{>}(x)=ax^{-\alpha}$ ($x>0$) if and only if its *point aggregate* $\Sigma_{p\in\Pi}p$ is governed by the Laplace transform of Eq. (3) (the exponent α being in the range $0<\alpha<1$).

Poissonian renormalization. The same as in the Fréchet case.

Poisson processes defined on the positive half-line and governed by power-law survival rate functions—which underlay both the aforementioned Fréchet and Lévy laws—are called *Paretian Poisson processes*, and turn out to play a focal role in statistical physics. See [18] for a detailed exposition of this important class of processes.

IV. NONLINEAR FRACTAL PROBABILITY LAWS

In this section we present the three *nonlinear* classes of fractal probability laws defined on the positive half-line: *hyper Pareto*, *hyper beta*, and *hyper shot noise*.

These fractal probability laws are one-dimensional projections of underlying Poissonian processes, which are the fixed points of power-law Poissonian renormalizations—a fact rendering them the classification "nonlinear." This nonlinearity, in turn, is precisely why these fractal distributions are unattainable via extreme value theory and via the central limit theorem (which consider only linear stochastic scaling limits).

Contrary to the fixed points of linear Poissonian renormalizations, the fixed points of power-law Poissonian renormalizations cannot range over the entire positive half-line $(0,\infty)$. Rather, they may range either on the unit interval $(0,1)$ or on the ray $(1,\infty)$. [Note that the scaling functions of power-law Poissonian renormalizations indeed map both the unit interval $(0,1)$ and the ray $(1,\infty)$ onto themselves].

A. Hyper Pareto

Definition. The hyper Pareto law is defined on the ray $(1,\infty)$ and governed by the survival probability function

$$\mathcal{P}_{>}(x) = \exp\{-a[\ln(x)]^{\alpha}\} \tag{5}$$

($x \geq 1$, the coefficient a and the exponent α being positive parameters).

In the special case $\alpha=1$ the hyper Pareto law reduces to a Pareto law governed by the survival probability function $\mathcal{P}_{>}(x)=x^{-a}$ ($x\geq 1$).

Power-law structure. Among the class of probability distributions defined on the ray $(1,\infty)$ the aforementioned Pareto law is the only probability distribution characterized by a power-law *survival probability function*.

Probabilistic renormalization. Among the class of probability distributions defined on the ray $(1,\infty)$ the aforementioned Pareto law is singled out by the following "conditional renormalization" characterization.

Let X be a random variable taking values in the ray $(1,\infty)$, and consider the scaled random variable X/l—having provided the information that X is greater than the level l ($l>1$). Then, the probability distribution of the scaled random variable X/l, *conditioned* on the information $\{X>l\}$, is independent of the level l if and only if the random variable X is governed by a power-law survival probability function of the form $\mathcal{P}_{>}(x)=x^{-a}$ ($x\geq 1$).

Poissonian structure. The hyper Pareto law is the probability distribution of the *minima* of Poisson processes defined on the ray $(1,\infty)$ and governed by *cumulative rate functions*, which are powers of logarithms. More specifically, a Poisson process Π defined on the ray $(1,\infty)$ is governed by the cumulative rate function $\mathcal{R}_{\leq}(x)=a[\ln(x)]^{\alpha}$ ($x>1$) if and only if its *minimal point* $\min_{p\in\Pi}\{p\}$ is governed by the survival probability function of Eq. (5).

Poissonian renormalization. Poisson processes defined on the ray $(1,\infty)$ and governed by cumulative rate functions, which are powers of logarithms, constitute the fixed points of *power-law Poissonian renormalizations* with positive exponents. More specifically, a Poisson process Π defined on the ray $(1,\infty)$ is a fixed point of a power-law Poissonian renormalization with scaling functions $\phi_k(x)=x^{k^{1/\alpha}}$ if and only if its cumulative rate function admits the form $\mathcal{R}_{\leq}(x)=a[\ln(x)]^{\alpha}$ ($x>1$).

B. Hyper beta

Definition. The hyper beta law is defined on the unit interval $(0,1)$ and governed by the cumulative distribution function

$$\mathcal{P}_{\leq}(x) = \exp\{-a[-\ln(x)]^{\alpha}\} \tag{6}$$

($0\leq x\leq 1$, the coefficient a and the exponent α being positive parameters).

In the special case $\alpha=1$ the hyper beta law reduces to a beta law governed by the cumulative distribution function $\mathcal{P}_{\leq}(x)=x^{a}$ ($0\leq x\leq 1$).

Power-law structure. Among the class of probability distributions defined on the unit interval the aforementioned beta law is the only probability distribution characterized by a power-law *cumulative distribution function*.

Probabilistic renormalization. Among the class of probability distributions defined on the unit interval the aforementioned beta law is singled out by the following "conditional renormalization" characterization.

Let X be a random variable taking values in the unit interval, and consider the scaled random variable X/l—having provided the information that X is smaller than the level l ($0<l<1$). Then, the probability distribution of the scaled random variable X/l, *conditioned* on the information $\{X<l\}$, is independent of the level l if and only if the random variable X is governed by a power-law cumulative distribution function of the form $\mathcal{P}_{\leq}(x)=x^{a}$ ($0\leq x\leq 1$).

Poissonian structure. The hyper beta law is the probability distribution of the *maxima* of Poisson processes defined on the unit interval and governed by *survival rate functions,* which are powers of logarithms. More specifically, a Poisson process Π defined on the unit interval is governed by the survival rate function $\mathcal{R}_{>}(x)=a[-\ln(x)]^{\alpha}$ ($0<x<1$) if and only if its *maximal point* $\max_{p\in\Pi}\{p\}$ is governed by the cumulative distribution function of Eq. (6).

Poissonian renormalization. Poisson processes defined on the unit interval and governed by survival rate functions, which are powers of logarithms, constitute the fixed points of *power-law Poissonian renormalizations* with positive exponents. More specifically, a Poisson process Π defined on the unit interval is a fixed point of a power-law Poissonian renormalization with scaling functions $\phi_k(x)=x^{k^{1/\alpha}}$ if and only if its survival rate function admits the form $\mathcal{R}_{>}(x)=a[-\ln(x)]^{\alpha}$ ($0<x<1$).

TABLE I. Summary of the properties of the six classes of fractal probability laws considered: (1) the range, or support, of the probability law; (2) the probability law's characteristic admitting a power-law structure; (3) the probabilistic renormalization under which the probability law is invariant; (4) the underlying Poisson process [characterized by either its cumulative rate function $\mathcal{R}_{\leq}(x)$ or by its survival rate function $\mathcal{R}_{>}(x)$]; (5) the one-dimensional projection linking the underlying Poisson process to the probability law; and (6) the Poissonian renormalization under which the underlying Poisson process is invariant. In the hyper Pareto and the hyper beta classes the inherent power-law structure and the probabilistic renormalization refer, respectively, to the Pareto and the beta subclasses.

	Range	Inherent Power-law structure	Probabilistic renormalization	Underlying Poissonian structure	Projection	Poissonian renormalization
Weibull	$(0,\infty)$	Forward hazard rate	Min renormalization	$\mathcal{R}_{\leq}(x)=ax^{\alpha}$ $(0<x<\infty)$	Minimal point	Linear
Fréchet	$(0,\infty)$	Backward hazard rate	Max renormalization	$\mathcal{R}_{>}(x)=ax^{-\alpha}$ $(0<x<\infty)$	Maximal point	Linear
Lévy	$(0,\infty)$	Log-Laplace transform	Sum renormalization	$\mathcal{R}_{>}(x)=ax^{-\alpha}$ $(0<x<\infty)$	Point aggregate	Linear
Hyper Pareto	$(1,\infty)$	Survival probability function	Conditional renormalization	$\mathcal{R}_{\leq}(x)=a[\ln(x)]^{\alpha}$ $(1<x<\infty)$	Minimal point	Power law
Hyper beta	(0,1)	Cumulative distribution function	Conditional renormalization	$\mathcal{R}_{>}(x)=a[-\ln(x)]^{\alpha}$ $(0<x<1)$	Maximal point	Power law
Hyper shot noise	$(0,\infty)$	Cumulant sequence		$\mathcal{R}_{>}(x)=a[-\ln(x)]^{\alpha}$ $(0<x<1)$	Point aggregate	Power law

C. Hyper shot noise

Definition. The hyper shot noise law is defined on the positive half-line $(0,\infty)$ and governed by the cumulant sequence

$$\mathcal{C}(m) = \Gamma(1+\alpha)am^{-\alpha} \tag{7}$$

($m=1,2,\ldots$, the coefficient a and the exponent α being positive parameters).

In the special case $\alpha=1$ the hyper shot noise law reduces to what we refer to as the "shot noise law": The stationary distribution of a *shot noise* process driven by a Poissonian noise [21–23] (see also Chapter 9 in [24] for a more recent account). Specifically, consider a shot noise process $[\xi(t)]_t$, with relaxation parameter κ, which is driven by a homogeneous Poisson process $[N(t)]_t$ with intensity λ. Then [24–26] (i) the dynamics of the shot noise process $[\xi(t)]_t$ are governed by the Ornstein-Uhlenbeck stochastic differential equation $\dot{\xi}(t)=-\kappa\xi(t)+\dot{N}(t)$; and (ii) the stationary distribution of the shot noise process $[\xi(t)]_t$ is governed by the *harmonic* cumulant sequence $\mathcal{C}(m)=am^{-1}$ $(m=1,2,\ldots)$ with "amplitude" $a=\lambda/\kappa$.

Power-law structure. Among the class of probability distributions defined on the positive half-line the hyper shot noise law is the only probability distribution characterized by a power-law *cumulant sequence*.

Poissonian structure. The hyper shot noise law is the probability distribution of the *point aggregates* of Poisson processes defined on the unit interval and governed by *survival rate functions* which are powers of logarithms. More specifically, a Poisson process Π defined on the unit interval is governed by the survival rate function $\mathcal{R}_{>}(x)=a[-\ln(x)]^{\alpha}$ $(0<x<1)$ if and only if its *point aggregate* $\Sigma_{p\in\Pi}p$ is governed by the cumulant sequence of Eq. (7).

Poissonian renormalization. The same as in the hyper beta case.

V. CONCLUSIONS

This research provides a panoramic view and a systematic classification of the notion of *fractality* in the context of positive-valued statistical distributions. We explored six classes of fractal probability laws defined on the positive half-line. Each of these classes features a different type of fractality, manifested by a power-law structure of a corresponding probabilistic characteristic. All classes are one-dimensional projections of underlying Poisson processes which, in turn, are the unique fixed points of Poissonian renormalizations: the *linear* fractal classes of Weibull, Fréchet, and Lévy corresponding to linear Poissonian renormalizations; the *nonlinear* fractal classes of hyper Pareto, hyper beta, and hyper shot noise corresponding to power-law Poissonian renormalizations. The properties of the six fractal probability laws are summarized in Table I.

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[2] V. Pareto, *Cours d'Économie Politique* (Droz, Geneva, 1896).
[3] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).
[4] P. Bak, *How Nature Works* (Copernicus, New York, 1996).
[5] *Scale invariance and beyond*, edited by B. Dubrulle, F. Garner, and D. Sornette (EDP Sciences and Springer, Berlin, 1997).
[6] J. Klafter and I. M. Sokolov, Phys. World **18**, 29 (2005).
[7] M. E. J. Newman, Contemp. Phys. **46**, 323 (2005).
[8] I. Eliazar and J. Klafter, Physica A **383**, 171 (2007).
[9] I. Eliazar and J. Klafter, e-print arXiv:0804.3300, Cond-Mat.Stat-Mech (to be published).
[10] E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, Classics in Applied Mathematics 17, reprint edition (Society for Industrial & Applied Mathematics, 1996).
[11] B. Gnedenko, Ann. Math. **44**, 423 (1943);translated and reprinted in *Breakthroughs in Statistics I*, edited by S. Kotz and N. L. Johnson (Springer, New York, 1992), pp.195–225.
[12] E. J. Gumbel, *Statistics of Extremes* (Columbia University Press, New York, 1958).
[13] J. Galambos, *Asymptotic Theory of Extreme Order Statistics*, 2nd ed. (Krieger, New York, 1987).
[14] P. Lévy, *Théorie de l'Addition des Variables Aléatoires* (Gauthier-Villars, Paris, 1954).
[15] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, London, 1954).
[16] I. A. Ibragimov and Yu. V. Linnik, *Independent and Stationary Sequences of Random Variables* (Walterss-Noordhoff, Groningen, The Netherlands, 1971).
[17] P. Embrechts, C. Kluppelberg, and T. Mikosch, *Modeling Extremal Events for Insurance and Finance* (Springer, New York, 1997).
[18] I. Eliazar and J. Klafter, J. Stat. Phys. **131**, 487 (2008).
[19] I. Eliazar and J. Klafter, Physica A **386**, 318 (2007).
[20] J. F. C. Kingman, *Poisson Processes* (Oxford University Press, Oxford, 1993).
[21] W. Schottky, Phys. Rev. **28**, 74 (1926).
[22] S. O. Rice, Bell Syst. Tech. J. **23**, 282 (1944).
[23] S. O. Rice, Bell Syst. Tech. J. **24**, 146 (1945).
[24] S. B. Lowen and M. C. Teich, *Fractal-Based Point Processes* (Wiley, New York, 2005).
[25] I. Eliazar and J. Klafter, J. Stat. Phys. **119**, 165 (2005).
[26] I. Eliazar and J. Klafter, Proc. Natl. Acad. Sci. U.S.A. **102**, 13779 (2005).